On the Capacity of Spatially Correlated MIMO Rayleigh-Fading Channels

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***Abstract*—In this paper, we investigate the capacity distribution of spatially correlated, multiple-input–multiple-output (MIMO) channels. In particular, we derive a concise closed-form expression for the characteristic function (c.f.) of MIMO system capacity with arbitrary correlation among the transmitting antennas or among the receiving antennas in frequency-flat Rayleigh-fading environments. Using the exact expression of the c.f., the probability density function (pdf) and the cumulative distribution function (CDF) can be easily obtained, thus enabling the exact evaluation of the outage and mean capacity of spatially correlated MIMO channels. Our results are valid for scenarios with the number of transmitting antennas greater than or equal to that of receiving antennas with arbitrary correlation among them. Moreover, the results are valid for an arbitrary number of transmitting and receiving antennas in uncorrelated MIMO channels. It is shown that the capacity loss is negligible even with a correlation coefficient between two adjacent antennas as large as 0.5 for exponential correlation model. Finally, we derive an exact expression for the mean value of the capacity for arbitrary correlation matrices.**

***Index Terms*—Eigenvalues distribution, multiple input–multiple output (MIMO), multiple antennas, Rayleigh-fading channels, Shannon capacity, Wishart matrices.**

I. Introduction

It has been recognized in recent years that the use of multiple transmitting and receiving antennas can potentially provide large spectral efficiency for wireless communications in the presence of multipath fading environments [1], [2]. The analysis of capacity distribution for multiple-input–multiple-output (MIMO) channels in [3], [4] suggested practical structures to obtain large spectral efficiency, leading to the Bell Laboratories layered space–time (BLAST) architecture and to space–time codes [5], [6].

These MIMO systems can be studied from two different perspectives: one concerns performance evaluation in terms of error probability of practical systems, the other concerns the evaluation of the information-theoretic (Shannon) capacity. The former can be obtained by simulation [7], [8] or analytically, as presented recently in [9], [10]. For the latter, the complementary cumulative distribution function (CCDF) of the capacity (sometimes called capacity outage) was studied by Monte Carlo simulation in [2]–[4], and the mean capacity was derived in [11] for uncorrelated MIMO Rayleigh-fading channels. The analysis of MIMO systems in block Rayleigh-fading channels is presented in [12], and a Gaussian approximation to the capacity distribution is investigated in [13]. All of these analyses showed that MIMO systems in uncorrelated Rayleigh-fading environments can potentially provide enormous Shannon capacities.

In many practical situations, however, signal correlation among the antenna elements exists in realistic environments due to poor scattering conditions. This has given an impetus for studying MIMO systems in correlated fading environments. Toward this end, the effect of signal correlation on MIMO systems have been recently studied by Monte Carlo simulation in [14]–[16].

In this paper, we solve the problem of analytically evaluating the capacity distribution. More precisely, we derived the characteristic function (c.f.) of the capacity for MIMO Rayleigh-fading channels in concise closed form with arbitrary correlation among the transmitting elements or among the receiving elements. This enables the analytical evaluation of the capacity in terms of probability density function (pdf), cumulative distribution function (CDF), and CCDF. Using the identities given in the Appendix, we also derive the exact expression of the mean value of the capacity.

In Section II, we present the system model and, in Section III, we investigate the distribution of the eigenvalues of a Wishart matrix with a given correlation. In Sections IV and V, we provide the exact compact expressions for the c.f. and the mean value of the capacity, respectively. In Section VI, we show some results and, in Section VII, we present a summary and conclusions. Finally, in the Appendix, we derive some useful identities for evaluation of multiple integrals.

II. MIMO System Model

The MIMO system investigated in this work consists of N_{T} transmitting and N_{R} receiving antennas. We consider the equivalent low-pass signals after matched filtering and sampling. Throughout the paper, vectors and matrices are indicated by bold, $|\boldsymbol{A}|$ and $\det \boldsymbol{A}$ denote the determinant of matrix $\boldsymbol{A}$, and $\{a_{i,j}\}_{i,j=1,\ldots,N}$ is an $N \times N$ matrix with elements $a_{i,j}$, $i, j = 1 \ldots N$. Also, $\mathbb{E}\{\cdot\}$ denotes expectation, and in particular $\mathbb{E}_X\{\cdot\}$ denotes expectation with respect to the random variable (r.v.) X. The superscript † denotes conjugation and transposition.

Manuscript received November 8, 2002; revised June 27, 2003. The work of M. Chiani and A. Zanella was supported in part by Ministero dell'Istruzione, dell'Università e della Ricerca (MIUR). The work of M. Z. Win was supported in part by the Charles Stark Draper Endowment. The material in this paper was presented in part at the International Symposium on Wireless Communications, Victoria, BC, Canada, September 2002.

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Communicated by T. L. Marzetta, Guest Editor.

Digital Object Identifier 10.1109/TIT.2003.817437

The N_{R}-dimensional signal $\boldsymbol{y}$ at the output of the receiving antennas in flat fading can be written as [2]–[4], [11]

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{x}$ is the N_{T}-dimensional transmitted vector with complex components, and $\boldsymbol{n}$ is an N_{R}-dimensional vector with zero-mean independent and identically distributed (i.i.d.) complex Gaussian entries with independent real and imaginary parts having equal variance. The channel matrix $\boldsymbol{H}$, defined by

$$\boldsymbol{H} \triangleq \begin{bmatrix} | & | & & | \\ \boldsymbol{h}_1 & \boldsymbol{h}_2 & \cdots & \boldsymbol{h}_{N_{\mathrm{T}}} \\ | & | & & | \end{bmatrix} \tag{2}$$

is an $(N_{\mathrm{R}} \times N_{\mathrm{T}})$ random matrix with complex elements $\{h_{i,j}\}$ describing the gain of the radio channel between the jth transmitting antenna and the ith receiving antenna. We denote the jth column of $\boldsymbol{H}$ by $\boldsymbol{h}_j$, i.e., the N_{R}-dimensional propagation vector corresponding to the jth transmitted signal.

For uncorrelated MIMO Rayleigh-fading channels, the entries of $\boldsymbol{H}$ are i.i.d. Gaussian r.v.'s with zero-mean, independent real and imaginary parts with equal variance. When the correlation among the receiving antennas exists, the columns of $\boldsymbol{H}$ are independent random vectors, but the elements of each column are correlated with the same mean and covariance matrix. For the case of Rayleigh fading, this implies that $\mathbb{E}\{\boldsymbol{h}_j\} = \boldsymbol{0}$ and correlation matrix indicated as $\boldsymbol{\Sigma} = \mathbb{E}\{\boldsymbol{h}_j\, \boldsymbol{h}_j^\dagger\}$ for $j = 1, \ldots, N_{\mathrm{T}}$. Without loss of generality, the diagonal elements of $\boldsymbol{\Sigma}$ can be normalized to 1, i.e., $\mathbb{E}\{|h_{i,j}|^2\} = 1$, where the expectation is with respect to Rayleigh fading. Similarly, correlation among the transmitting antennas can be considered. In this case, the rows of $\boldsymbol{H}$ are independent, but the elements of each row are correlated with a given covariance matrix (the same for all rows).

In this paper, we consider the scenario in which the transmitter has no channel state information (CSI). In this case, each antenna transmits an average power P/N_{T}; thus the total transmitted power is $\mathbb{E}\{\boldsymbol{x}^\dagger \boldsymbol{x}\} = P$. The capacity of MIMO channels when the transmitter has no CSI is given by

$$C = \log_2 \det\left(\boldsymbol{I} + \frac{\rho}{N_{\mathrm{T}}} \boldsymbol{H}\boldsymbol{H}^\dagger\right) \qquad \text{(bit/s/Hz)} \tag{3}$$

where ρ is the average signal-to-noise ratio (SNR) per receiving antenna and $\boldsymbol{I}$ is the identity matrix [3], [4], [11]. Note that the capacity can be also written in terms of nonzero eigenvalues of the matrix $\boldsymbol{H}\boldsymbol{H}^\dagger$, since the determinant of $\boldsymbol{I} + \frac{\rho}{N_{\mathrm{T}}}\boldsymbol{H}\boldsymbol{H}^\dagger$ is the product of the eigenvalues and the zero eigenvalues of $\boldsymbol{H}\boldsymbol{H}^\dagger$ do not contribute to the product.

We now recall that, for any given two matrices $\boldsymbol{A}$ $(m \times p)$ and $\boldsymbol{B}$ $(p \times m)$ with $m \leq p$, the $(p \times p)$ matrix $\boldsymbol{B}\boldsymbol{A}$ has the same m eigenvalues as the $(m \times m)$ matrix $\boldsymbol{A}\boldsymbol{B}$, counting multiplicity, together with an additional $p - m$ eigenvalues identically equal to zero [17, p. 53]. Hence, nonzero eigenvalues of the matrices $\boldsymbol{H}^\dagger \boldsymbol{H}$ and $\boldsymbol{H}\boldsymbol{H}^\dagger$ are identical. Since this is true for every instantiation $\boldsymbol{H}$, the respective nonzero eigenvalues are equal in Law. This implies that the statistical distribution of the nonzero eigenvalues of $\boldsymbol{H}^\dagger \boldsymbol{H}$ and $\boldsymbol{H}\boldsymbol{H}^\dagger$ are equal.

Let $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_{N_{\min}}]^T$ denote the nonzero eigenvalues of the $(N_{\min} \times N_{\min})$ matrix $\boldsymbol{W}$, with $N_{\min} = \min\{N_{\mathrm{T}}, N_{\mathrm{R}}\}$ defined as

$$\boldsymbol{W} = \begin{cases} \boldsymbol{H}^\dagger \boldsymbol{H}, & \text{if } N_{\mathrm{R}} > N_{\mathrm{T}} \\ \boldsymbol{H}\boldsymbol{H}^\dagger, & \text{if } N_{\mathrm{R}} \leq N_{\mathrm{T}}. \end{cases} \tag{4}$$

Then, the capacity (3) can be written as [2], [11]

$$C = \sum_{i=1}^{N_{\min}} \log_2\left(1 + \frac{\rho}{N_{\mathrm{T}}}\lambda_i\right). \tag{5}$$

Since $\boldsymbol{H}$ is randomly varying, C is also randomly varying. The mean value of C for uncorrelated MIMO Rayleigh-fading channels is evaluated in [11], whereas the outage capacity (the CCDF of C) is investigated by Monte Carlo simulation in [2]–[4].

In the following sections, we will derive the c.f. of the capacity given in (5) for $\boldsymbol{H}$ with zero-mean complex Gaussian entries. We consider both cases with and without correlation among the antenna elements.

III. Distribution of the Eigenvalues of $\boldsymbol{W}$ for the MIMO Rayleigh-Fading Channel

Let us start by studying the matrix $\boldsymbol{W}$ in (4). When the elements of $\boldsymbol{H}$ are zero-mean complex Gaussian, $\boldsymbol{W}$ is called a central Wishart matrix. Wishart matrices are of great importance in multivariate statistical theory [18]–[21]. We will consider the uncorrelated and correlated cases separately in the following. The contribution of this section is to obtain the expressions for the joint pdf of the eigenvalues in terms of the product of determinants which are useful for analyzing MIMO systems.

A. The Uncorrelated Case

We first consider the scenario with uncorrelated fading among the antenna elements, a situation that arises when the antenna elements are spaced sufficiently far apart from each other.

The distribution of the ordered eigenvalues of a complex Wishart matrix is studied in [21]. The joint pdf of the ordered eigenvalues $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{N_{\min}}$ of $\boldsymbol{W}$ is [22]

$$f_{\boldsymbol{\lambda}}(x_1, \ldots, x_{N_{\min}}) = K \prod_{i=1}^{N_{\min}} e^{-x_i} x_i^{N_{\max} - N_{\min}} \cdot \prod_{i<j}^{N_{\min}} (x_i - x_j)^2 \tag{6}$$

where K is a normalizing constant given by

$$K = \frac{\pi^{N_{\min}(N_{\min}-1)}}{\tilde{\Gamma}_{N_{\min}}(N_{\max})\tilde{\Gamma}_{N_{\min}}(N_{\min})} \tag{7}$$

with

$$\tilde{\Gamma}_{N_{\min}}(n) = \pi^{N_{\min}(N_{\min}-1)/2} \prod_{i=1}^{N_{\min}} (n-i)! \tag{8}$$

and $N_{\max} = \max\{N_{\mathrm{T}}, N_{\mathrm{R}}\}$.

Denoting $\boldsymbol{x} = [x_1, x_2, \ldots, x_{N_{\min}}]$, the pdf in (6) can be written alternatively, in terms of the Vandermonde matrix $\boldsymbol{V}_1(\boldsymbol{x})$ defined by

$$\boldsymbol{V}_1(\boldsymbol{x}) \triangleq \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{N_{\min}} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{N_{\min}-1} & x_2^{N_{\min}-1} & \cdots & x_{N_{\min}}^{N_{\min}-1} \end{bmatrix}. \quad (9)$$

Since $|\boldsymbol{V}_1(\boldsymbol{x})|^2 = \prod_{i<j}^{N_{\min}} (x_i - x_j)^2$ [17, p. 29], (6) becomes

$$f_{\boldsymbol{\lambda}}(x_1, \ldots, x_{N_{\min}}) = K|\boldsymbol{V}_1(\boldsymbol{x})|^2 \prod_{i=1}^{N_{\min}} e^{-x_i} x_i^{N_{\max}-N_{\min}}. \quad (10)$$

Note that the expression (10) is valid for arbitrary N_{T} and N_{R}.

B. The Correlated Case

Let us now consider a MIMO system in Rayleigh-fading channels with uncorrelated signals at the transmitting antennas, but with correlated signals at the receiving antennas, characterized by a given correlation matrix $\boldsymbol{\Sigma}$. A typical example of this is a downlink transmission from a base station (BS) to mobile station (MS), where the antennas at the BS can be spaced sufficiently far enough to achieve uncorrelation among them. On the other hand, it is more difficult to space the antennas far apart at the mobile terminals due to physical size constraints, and consequently correlation arises among the antenna elements in such scenarios. The dual case of correlation at the transmitter will be discussed at the end of this section.

In studying the scenarios with correlation among the receiving antennas, we consider the case $N_{\mathrm{R}} \leq N_{\mathrm{T}}$ so that $\boldsymbol{W} = \boldsymbol{H}\boldsymbol{H}^{\dagger}$ is a full rank (with probability 1) central Wishart matrix. In this case, the joint pdf of the (real) ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N_{\min}}$ of $\boldsymbol{W}$ is given in [21] by

$$f_{\boldsymbol{\lambda}}(x_1, \ldots, x_{N_{\min}}) = K|\boldsymbol{\Sigma}|^{-N_{\max}} {}_0\tilde{F}_0\left(-\boldsymbol{\Sigma}^{-1}, \boldsymbol{W}\right) \cdot |\boldsymbol{W}|^{N_{\max}-N_{\min}} \cdot \prod_{i<j}^{N_{\min}} (x_i - x_j)^2 \quad (11)$$

where ${}_0\tilde{F}_0(\boldsymbol{A}, \boldsymbol{B})$ is known as the hypergeometric function of Hermitian matrix arguments, whose definition is given in [21, eq. (88)] in terms of a series involving *zonal polynomials*. These polynomials are in general very difficult to manage, and the form of (11) does not lend itself into a tractable form for further analysis.

It is desirable to obtain a friendlier expression for the joint pdf of $\lambda_1, \lambda_2, \ldots, \lambda_{N_{\min}}$ that is useful for analyzing MIMO systems. In the rest of the section, we derive an alternative expression for the joint pdf that is amenable to further analysis. Due to the definitions of zonal polynomial and of hypergeometric functions of Hermitian matrices [21, eq. (85)] and [21, eq. (88)], we also have

$${}_0\tilde{F}_0(\boldsymbol{A}, \boldsymbol{B}) = {}_0\tilde{F}_0(D_{\boldsymbol{A}}, D_{\boldsymbol{B}}) \quad (12)$$

where $D_{\boldsymbol{A}}$ and $D_{\boldsymbol{B}}$ are diagonal matrices whose diagonal elements are eigenvalues of the $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. Let $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \ldots, \sigma_{N_{\min}}]$, with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{N_{\min}} \geq 0$ denoting the ordered eigenvalues of $\boldsymbol{\Sigma}$. When these eigenvalues are all distinct, ${}_0\tilde{F}_0(-\boldsymbol{\Sigma}^{-1}, \boldsymbol{W})$ can be expressed, using (12) together with [23, Lemma 3], in terms of determinants of matrices whose elements are hypergeometric functions of scalar arguments. In particular[1]

$${}_0\tilde{F}_0\left(-\boldsymbol{\Sigma}^{-1}, \boldsymbol{W}\right) = \zeta_{N_{\min}} \frac{|\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\sigma})|}{|\boldsymbol{V}_1(\boldsymbol{x})|\,|\boldsymbol{V}_2(\boldsymbol{\sigma})|} \quad (13)$$

where $\zeta_{N_{\min}}$ is a constant defined as

$$\zeta_{N_{\min}} \triangleq \prod_{j=1}^{N_{\min}} (j-1)! \quad (14)$$

$\boldsymbol{V}_2(\boldsymbol{\sigma})$ is a Vandermonde matrix given by

$$\boldsymbol{V}_2(\boldsymbol{\sigma}) \triangleq \boldsymbol{V}_1\left(-\left[\sigma_1^{-1}, \ldots, \sigma_{N_{\min}}^{-1}\right]\right) \quad (15)$$

and $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\sigma})$ is defined by (16) at the bottom of the page.

Now, by recalling that ${}_0\tilde{F}_0(y) = e^y$ [25], substituting (13) in (11), and expressing the determinants in (11) as product of eigenvalues, we obtain an alternative expression for the joint pdf of $\lambda_1, \lambda_2, \ldots, \lambda_{N_{\min}}$ as

$$f_{\boldsymbol{\lambda}}(x_1, \ldots, x_{N_{\min}}) = K_{\boldsymbol{\Sigma}}|\boldsymbol{E}(\boldsymbol{x}, \boldsymbol{\sigma})| \cdot |\boldsymbol{V}_1(\boldsymbol{x})| \cdot \prod_{j=1}^{N_{\min}} x_j^{N_{\max}-N_{\min}} \quad (17)$$

where $K_{\boldsymbol{\Sigma}}$, a normalizing constant, depends only on the correlation matrix $\boldsymbol{\Sigma}$ through its eigenvalues, given by

$$K_{\boldsymbol{\Sigma}} = K \cdot \zeta_{N_{\min}} \cdot \frac{|\boldsymbol{\Sigma}|^{-N_{\max}}}{|\boldsymbol{V}_2(\boldsymbol{\sigma})|} \quad (18)$$

[1] Note that an equation similar to (13) was used for the ${}_1\tilde{F}_0(.;\,.,\,.)$ to analyze minimum mean-square error (MMSE) combining in [24, eq. (6)].

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\sigma}) \triangleq \begin{bmatrix} {}_0\tilde{F}_0(-x_1/\sigma_1) & {}_0\tilde{F}_0(-x_2/\sigma_1) & \cdots & {}_0\tilde{F}_0(-x_{N_{\min}}/\sigma_1) \\ {}_0\tilde{F}_0(-x_1/\sigma_2) & {}_0\tilde{F}_0(-x_2/\sigma_2) & \cdots & {}_0\tilde{F}_0(x_{N_{\min}}/\sigma_2) \\ \vdots & \vdots & \ddots & \vdots \\ {}_0\tilde{F}_0(-x_1/\sigma_{N_{\min}}) & {}_0\tilde{F}_0(-x_2/\sigma_{N_{\min}}) & \cdots & {}_0\tilde{F}_0(-x_{N_{\min}}/\sigma_{N_{\min}}) \end{bmatrix}. \quad (16)$$

and $\boldsymbol{E}(\boldsymbol{x}, \boldsymbol{\sigma})$ is defined by

$$\boldsymbol{E}(\boldsymbol{x}, \boldsymbol{\sigma}) \triangleq \begin{bmatrix} e^{-\frac{x_1}{\sigma_1}} & e^{-\frac{x_2}{\sigma_1}} & \cdots & e^{-\frac{x_{N_{\min}}}{\sigma_1}} \\ e^{-\frac{x_1}{\sigma_2}} & e^{-\frac{x_2}{\sigma_2}} & \cdots & e^{-\frac{x_{N_{\min}}}{\sigma_2}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\frac{x_1}{\sigma_{N_{\min}}}} & e^{-\frac{x_2}{\sigma_{N_{\min}}}} & \cdots & e^{-\frac{x_{N_{\min}}}{\sigma_{N_{\min}}}} \end{bmatrix}. \quad (19)$$

Note that the new expression for joint pdf involves a product of determinants, and it will be apparent shortly that the form of (17) lends itself into a tractable form for further analysis.

It is worth noting that (17) requires a correlation matrix $\boldsymbol{\Sigma}$ with all distinct eigenvalues. The other extreme case is the uncorrelated scenario treated in the previous subsection, where all eigenvalues of $\boldsymbol{\Sigma}$ are identical. The intermediate cases where some eigenvalues of $\boldsymbol{\Sigma}$ are equal can be obtained as limiting cases of (17). Numerically, it is sufficient to slightly perturb the eigenvalues of $\boldsymbol{\Sigma}$ since all functions are continuous and eigenvalues of $\boldsymbol{\Sigma}$ are deterministic.

For the dual case of correlation at the transmitter side only, using the fact that $\boldsymbol{H}\boldsymbol{H}^\dagger$ and $\boldsymbol{H}^\dagger\boldsymbol{H}$ have the same nonzero eigenvalues, we can state the following [17].

Theorem 1 (Duality Theorem): The capacity of MIMO systems with $N_{\mathrm{T}} = N_1$ and $N_{\mathrm{R}} = N_2$, operating at $\rho = \rho_1$ in a Rayleigh-fading environment with correlation among the transmitting antennas, characterized by correlation matrix $\boldsymbol{\Sigma}$ for the rows of the channel matrix $\boldsymbol{H}$, is equal to the capacity of MIMO systems with $N_{\mathrm{T}} = N_2$ and $N_{\mathrm{R}} = N_1$, operating at $\rho = \rho_1 N_2/N_1$ in a Rayleigh-fading environment with correlation among the receiving antennas, characterized by the correlation matrix $\boldsymbol{\Sigma}$ for the columns of the channel matrix $\boldsymbol{H}$.

By using the Theorem 1, the results for MIMO systems with correlation among the transmitting antennas can be obtained from the results for the case with correlation among the receiving antennas. Hence, we will only consider the latter case in the following. When correlation is present at both ends, our results (neglecting the correlation on one side) are to be considered as upper bounds on the capacity.

IV. Exact Expression of the Characteristic Function of C

In this section, we will derive the c.f. of capacity using the joint pdf of the eigenvalues given in (10) and (17). By introducing the function

$$\varphi(x) = \left(1 + \frac{\rho}{N_{\mathrm{T}}} x\right)^{\frac{\jmath 2\pi z}{\ln 2}} \quad (20)$$

the c.f. of the capacity is written as

$$\begin{aligned} \phi_C(z) &\triangleq \mathbb{E}_C\left\{e^{\jmath 2\pi C z}\right\} \\ &= \mathbb{E}_{\boldsymbol{\lambda}}\left\{\prod_{i=1}^{N_{\min}} \varphi(\lambda_i)\right\} \quad (21) \\ &= \int \cdots \int_{\mathcal{D}_{\mathrm{ord}}} f_{\boldsymbol{\lambda}}(x_1, \ldots, x_{N_{\min}}) \prod_{i=1}^{N_{\min}} \varphi(x_i)\, d\boldsymbol{x} \quad (22) \end{aligned}$$

where the multiple integral is over the domain

$$\mathcal{D}_{\mathrm{ord}} = \{\infty > x_1 \geq x_2 \geq \cdots \geq x_{N_{\min}} \geq 0\}$$

and $d\boldsymbol{x} = dx_1\, dx_2 \cdots dx_{N_{\min}}$. Now the problem in (22) is the evaluation of the average of the product of a function $\varphi(\cdot)$ applied to the different eigenvalues of a Wishart matrix, where the average is taken with respect to eigenvalues distribution. Recall that in both the uncorrelated (10) and correlated (17) cases, the joint pdf's of the ordered eigenvalues of $\boldsymbol{W}$ are proportional to the product of determinants of matrices.

For the uncorrelated MIMO Rayleigh-fading channels, the eigenvalues distribution is given by (10). Applying Corollary 2 in the Appendix with

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{V}_1(\boldsymbol{x}) \quad (23)$$
$$\boldsymbol{\Psi}(\boldsymbol{x}) = \boldsymbol{V}_1(\boldsymbol{x}) \quad (24)$$

and $\xi(x) = x^{N_{\max}-N_{\min}} e^{-x} \varphi(x)$, the c.f. of the capacity reduces to the following compact expression:

$$\phi_C(z) = K \det \boldsymbol{U} \quad (25)$$

where $\boldsymbol{U}$ is an $N_{\min} \times N_{\min}$ Hankel matrix with ijth elements given by

$$u_{i,j} = \int_0^\infty x^{N_{\max}-N_{\min}+j+i-2} e^{-x} \varphi(x)\, dx. \quad (26)$$

For the correlated MIMO Rayleigh-fading channels, the eigenvalues distribution is given in (17). Again, by using Corollary 2 in the Appendix with

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{E}(\boldsymbol{x}, \boldsymbol{\sigma}) \quad (27)$$
$$\boldsymbol{\Psi}(\boldsymbol{x}) = \boldsymbol{V}_1(\boldsymbol{x}) \quad (28)$$

and $\xi(x) = x^{N_{\max}-N_{\min}} \varphi(x)$, the expected value of the product $\prod_{i=1}^{N_{\min}} \varphi(\lambda_i)$ reduced to the following compact expression:

$$\phi_C(z) = K_{\boldsymbol{\Sigma}} \det \boldsymbol{G} \quad (29)$$

where $\boldsymbol{G}$ is an $N_{\min} \times N_{\min}$ matrix with ijth elements given by

$$g_{i,j} = \int_0^\infty x^{N_{\max}-N_{\min}+j-1} e^{-x/\sigma_i} \varphi(x)\, dx. \quad (30)$$

In both cases, the functions in (26) and (30) can be evaluated in a compact closed form using the identity

$$\begin{aligned} &\int_0^\infty x^n (1+bx)^y e^{-\frac{x}{a}}\, dx \\ &\quad = \frac{n!\Gamma(-1-n-y)\, {}_1F_1(1+n, n+y+2, 1/ab)}{b^{n+1}\Gamma(-y)} \\ &\qquad + a^{n+1+y} b^y \Gamma(n+1+y)\, {}_1F_1(-y, -n-y, 1/ab) \quad (31) \end{aligned}$$

valid for $\Re\{a\} > 0$, $n \geq 0$, $\arg\{b\} \neq \pi$, where $\Gamma(\cdot)$ is the Gamma function and ${}_1F_1(\cdot, \cdot, \cdot)$ is the hypergeometric function [26].

Therefore, the c.f. of the capacity for the Rayleigh MIMO channel is written in concise closed form as the determinant of a matrix, as indicated by (25) and (29); these are the key contributions of this paper. Due to its simplicity, from the c.f.

all other distribution functions (pdf and CDF) can be simply obtained by means of a single integral, that can be evaluated efficiently for instance by fast Fourier transform (FFT) methods.

To give an example, we provide the expression for the CDF of the capacity

$$F_C(x) = \int_{-\infty}^{\infty} \phi_C(z) \left[\frac{1 - e^{-\jmath 2\pi z x}}{\jmath 2\pi z} \right] dz \tag{32}$$

where we used the fact that the random variable C is not negative.

V. Mean Capacity for Correlated MIMO Systems

Here we derive the mean capacity expression for correlated MIMO systems, by using an identity from the Appendix. From (5), the mean capacity can be written as

$$\mathbb{E}\{C\} = \mathbb{E}\left\{ \sum_{i=1}^{N_{\min}} \log_2 \left(1 + \frac{\rho}{N_{\rm T}} \lambda_i\right)\right\} \tag{33}$$

that can be interpreted as the mean of the sum of a given function applied to the different eigenvalues of $\boldsymbol{W}$. Thus, starting from (17) and using Theorem 3 in the Appendix with

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{E}(\boldsymbol{x}, \boldsymbol{\sigma}) \tag{34}$$

$$\boldsymbol{\Psi}(\boldsymbol{x}) = \boldsymbol{V}_1(\boldsymbol{x}) \tag{35}$$

$\xi(x) = x^{N_{\max} - N_{\min}}$ and $\tilde{\xi}(x) = \log_2(1 + \rho x / N_{\rm T})$, we immediately obtain the following expression:

$$\mathbb{E}\{C\} = K_{\boldsymbol{\Sigma}} \sum_{k=1}^{N_{\min}} \det \left\{ \int_0^{\infty} x^{N_{\max} - N_{\min} + j - 1} \cdot e^{-x/\sigma_i} U_{k,j}\left(\log_2 \left(1 + \frac{\rho}{N_{\rm T}} x \right) \right) dx \right\}_{i,j=1,\ldots,N_{\min}} \tag{36}$$

where σ_i are the eigenvalues of the correlation matrix, and $U_{k,j}(x)$ is defined in (54).

The expression (36) thus extends the result of [11, eq. (8)] on the mean capacity to the case of spatially correlated MIMO Rayleigh-fading channels. Moreover, by differentiating the c.f. in (25) and (29) using well-known rules for the derivative of the determinant of a matrix [27, eq. (6.5.9)], all moments of the r.v. C can be also easily derived.

VI. Analysis of Some Correlated Scenarios

The analytical framework we derived is general and valid for arbitrary correlation matrices $\boldsymbol{\Sigma}$. To give an example, we consider two well-known correlation models: exponential correlation with $\boldsymbol{\Sigma} = \{r^{|i-j|}\}_{i,j=1,\ldots,N_{\rm R}}$ and $r \in [0, 1)$ [28]; and a recent model proposed in [15]. In the latter model, in case of Rayleigh-fading environment, the generic element of $\boldsymbol{\Sigma}$ becomes

$$\mathcal{I}_0\left(\sqrt{\eta^2 - 4\pi^2 d_{ij}^2 + \jmath 4\pi\eta \sin(\mu) d_{ij}} \right) / \mathcal{I}_0(\eta)$$

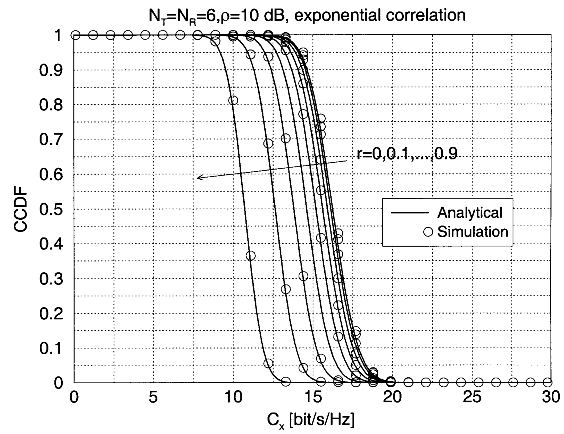

Fig. 1. CCDF of the capacity for a MIMO system with $N_{\rm T} = 6$, $N_{\rm R} = 6$, and SNR per receiving antenna $\rho = 10$ dB. Exponential correlation case with r ranging from 0 to 0.9.

with $i, j = 1, \ldots, N_{\rm R}$ [15, eq. (27)]. The parameter η controls the width of the angle-of-arrival (AOA), ranging from 0 (isotropic scattering) to ∞ (extremely nonisotropic scattering), $\mu \in [\pi, \pi)$ accounts for the mean direction of the AOA, d_{ij} is the distance (normalized with respect to the wavelength) between elements i and j of the receiving antenna array, and $\mathcal{I}_0(\cdot)$ is the zero-order modified Bessel function.

A. Exponential Correlation Model

Fig. 1 shows the CCDF of the capacity, which gives the probability that C is larger than the abscissa C_x, for a MIMO system with $N_{\rm T} = 6$, $N_{\rm R} = 6$; the SNR per receiving antenna is fixed to $\rho = 10$ dB and the parameter r ranges from 0 to 0.9. The figure shows that the capacity reduction is negligible for small values of r, but it becomes significant for $r > 0.5$. It can be seen that we have a 90% probability that the capacity is larger than 13 bits/s/Hz at $r = 0.5$ and reduces to 9 bits/s/Hz at $r = 0.9$. We also compare our exact analytical expressions with Monte Carlo simulations; the latter are carried out by generating 10 000 realizations of $\boldsymbol{H}$ and evaluating (3). As expected, the comparison shows an excellent agreement between analysis and simulation.

It can be concluded from Fig. 1 that, for this exponential correlation model, the effect of correlation is negligible when the maximum correlation between pairs of antenna elements r is less than 0.5. This result is in agreement with previous results on the effect of spatial correlation [15], [16], [29].

We next investigate effects of SNR and the number of antennas on the CCDF of the capacity. The CCDF of the capacity for various values of ρ is plotted in Fig. 2, for $N_{\rm T} = N_{\rm R} = 5$, $r = 0.7$, with ρ ranging from 5 to 30 dB. If we fix again 90% probability, the value of C_x we obtain ranges from about 6 bits/s/Hz for $\rho = 5$ dB to about 37 bits/s/Hz for $\rho = 30$ dB. The CCDF of the capacity for various values of $N_{\rm T} = N_{\rm R}$ is plotted in Fig. 3 for $\rho = 10$ dB, and $r = 0.8$. The figure shows that at 90% probability, C_x is larger than 3 bits/s/Hz when $N_{\rm T} = N_{\rm R} = 2$, and becomes larger than 19 bits/s/Hz when $N_{\rm T} = N_{\rm R} = 10$. Even with r as high as 0.8, it can be observed that the capacity increase is almost linear with $N_{\rm T} = N_{\rm R}$, as in the case for the capacity of uncorrelated MIMO systems [3], [4].

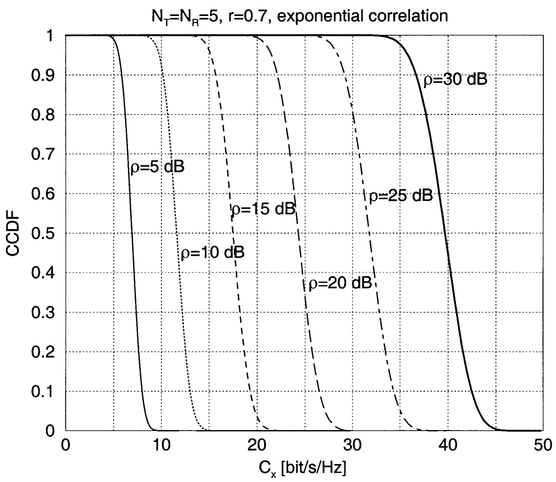

Fig. 2. CCDF of the capacity for a MIMO system with $N_{\mathrm{T}} = 5$, $N_{\mathrm{R}} = 5$, and $\rho = 5$ to 30 dB. Exponential correlation case with $r = 0.7$.

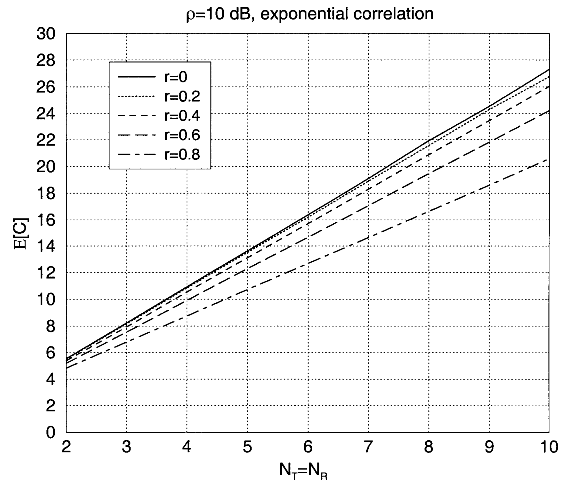

Fig. 4. Mean value of the capacity as a function of $N_{\mathrm{R}} = N_{\mathrm{T}}$ for a MIMO system with $\rho = 10$ dB. Exponential correlation case with r ranging from 0 to 0.8.

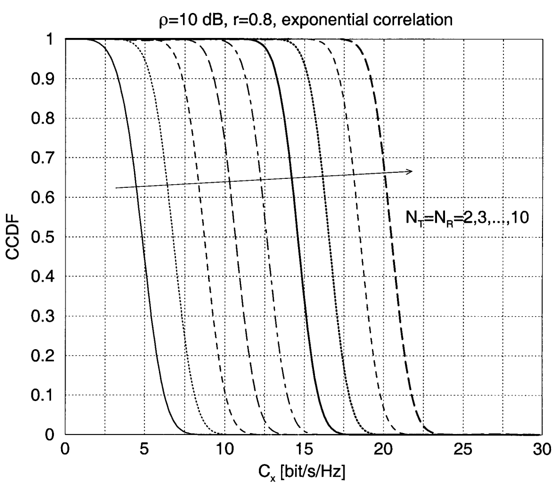

Fig. 3. CCDF of the capacity for a MIMO system with $\rho = 10$ dB. Exponential correlation case with $r = 0.8$, and $N_{\mathrm{T}} = N_{\mathrm{R}} = 2$ to 10.

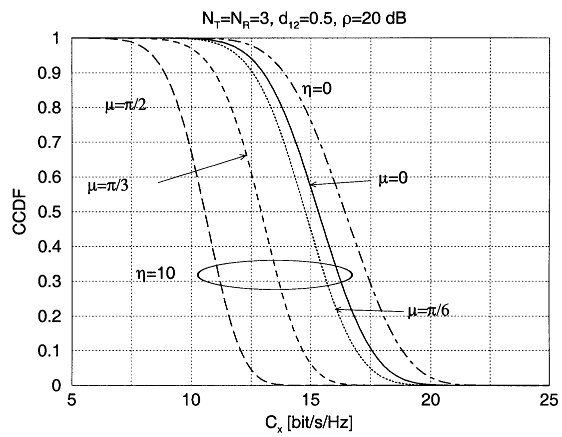

Fig. 5. CCDF of the capacity for a MIMO system with $N_{\mathrm{T}} = N_{\mathrm{R}} = 4$, and $\rho = 20$ dB. Correlation model of [15] with $d_{12} = 0.5$, $\eta = 0, 10$, and μ ranging from 0 to $\pi/2$.

Finally, Fig. 4 shows the mean capacity obtained by (36) as a function of $N_{\mathrm{T}} = N_{\mathrm{R}}$, for $\rho = 10$ dB and different values of r. The figure shows that the mean capacity increases almost linearly with the number of antennas; the presence of exponentialcorrelation among the receiving antennas only affects the slope of the curves. Furthermore, the reduction of capacity due to the correlation is negligible for values of r smaller than 0.4. On the other hand, when $r = 0.8$ and six antennas are considered, the reduction in terms of $\mathbb{E}\{C\}$ compared to the uncorrelated case is almost 4 bits/s/Hz.

B. Correlation Model of [15]

In this subsection, we show some example of the CCDF of the capacity for a MIMO system with the correlation model proposed in [15]. Here, we consider a linear array at the receiver with equally spaced antenna elements.

In Fig. 5, we have fixed $\rho = 20$ dB, $N_{\mathrm{T}} = N_{\mathrm{R}} = 3$, and the normalized (with respect to the wavelength) distance between the two adjacent antenna elements is 0.5; two values of η are considered, 0 (isotropic scattering) and 10, with μ as a parameter ranging from 0 to $\pi/2$. Note that, in the case of $\eta = 0$, the curves for different values of μ coincide (this can be easily derived by [15, eq. (27)]). When the propagation conditions range from an isotropic $(\eta = 0)$ to a nonisotropic $(\eta = 10)$ scattering, the capacity decreases; in particular, the reduction is significant when μ (the mean direction of the AOA) approaches $\pi/2$.

Finally, Fig. 6 shows the effect of the parameter η on the capacity; $N_{\mathrm{T}} = N_{\mathrm{R}} = 4$, $\rho = 20$ dB, $\mu = \pi/2$, and $d_{12} = 0.5$. The figure clearly shows that η, which controls the width of the AOA, has a strong influence on the capacity; if we fix CCDF $= 0.9$, the reduction in terms of C_x, as η ranges from 0 (isotropic scattering) to 7, is of about 7.5 bits/s/Hz.

VII. Conclusion

In this paper, we have derived a concise closed-form expression for the distribution in terms of c.f. of the capacity for multiple-antenna systems in frequency-flat, correlated, Rayleigh-fading channels. The analytical methodology we propose is general and can be used for arbitrary correlation on one side (transmitter or receiver). The proposed analysis allows a fast evaluation of the capacity pdf and of the outage capacity for MIMO systems. Moreover, the exact expression for the mean value of

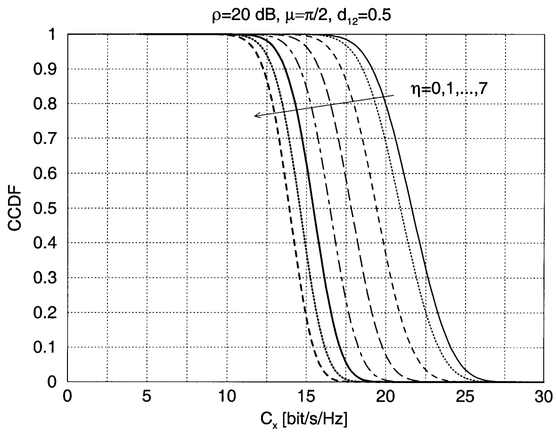

Fig. 6. CCDF of the capacity for a MIMO system with $N_{\rm T} = N_{\rm R} = 4$, $\rho = 20$ dB, $\mu = \pi/2$, and $d_{12} = 0.5$. Correction model of [15] with η ranging from 0 to 7.

the capacity has been derived; this result is a generalization of that provided in [11] for the uncorrelated case. Finally, numerical results show that, in case of an exponential correlation model, when the correlation coefficient between adjacent antenna elements is smaller than 0.5, the reduction in terms of capacity is negligible.

Appendix
Some Useful Identities

Let us start by recalling some basic results from linear algebra. The definition for the determinant of a generic matrix $\boldsymbol{A} = \{a_{i,j}\}_{i,j=1\ldots N}$ is given by [30]

$$|\boldsymbol{A}| = \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{i=1}^{N} a_{\sigma_i, i} \tag{37}$$

where $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_N$ is a permutation of the integers $1, \ldots, N$, the sum is over all permutations, and $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation. Using (37), we obtain a slightly different expression that is more useful for our purposes. We first note that permuting columns or rows of a matrix changes the sign of the determinant according to the sign of the permutation; we obtain an alternative expression for the determinant of a matrix as

$$|\boldsymbol{A}| = \mathrm{sgn}(\mu) \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{i=1}^{N} a_{\sigma_i, \mu_i} \tag{38}$$

where μ is an arbitrary permutation.

We now give an extension of this definition to rank 3 tensors, i.e., three-dimensional matrices.

Definition 1: Given a rank 3 tensor

$$\boldsymbol{A} = \{a_{i,j,k}\}_{i,j,k=1,\ldots,N}$$

we define the operator $\mathcal{T}(\boldsymbol{A})$ as

$$\mathcal{T}(\boldsymbol{A}) \triangleq \sum_{\mu} \mathrm{sgn}(\mu) \sum_{\alpha} \mathrm{sgn}(\alpha) \prod_{k=1}^{N} a_{\mu_k, \alpha_k, k} \tag{39}$$

where the sums are over all possible permutations μ, and α of the integers $1, \ldots, N$.

Note that when $a_{i,j,k}$ are independent of k, i.e., $a_{i,j,k} = a_{i,j,1}$, we have

$$\begin{aligned}\mathcal{T}(\boldsymbol{A}) &= \sum_{\mu} \mathrm{sgn}(\mu) \sum_{\alpha} \mathrm{sgn}(\alpha) \prod_{k=1}^{N} a_{\mu_k, \alpha_k, k} \\ &= N! \det\left(\{a_{i,j,1}\}_{i,j=1,\ldots,N}\right)\end{aligned} \tag{40}$$

i.e., the $\mathcal{T}(\cdot)$ degenerates into $N!$ times the determinant of $\{a_{i,j,1}\}_{i,j=1\ldots,N}$. Using the above definition, we give a useful theorem.

Theorem 2: Given two arbitrary $N \times N$ matrices $\boldsymbol{\Phi}(\boldsymbol{x})$ and $\boldsymbol{\Psi}(\boldsymbol{x})$ with ijth elements $\Phi_i(x_j)$ and $\Psi_i(x_j)$, and arbitrary functions $\xi_i(\cdot)$, the following identity holds:

$$\begin{aligned}&\int \cdots \int_{\mathcal{D}} |\boldsymbol{\Phi}(\boldsymbol{x})| \cdot |\boldsymbol{\Psi}(\boldsymbol{x})| \prod_{k=1}^{N} \xi_k(x_k)\, d\boldsymbol{x} \\ &\quad = \mathcal{T}\left(\left\{\int_a^b \Phi_i(x)\Psi_j(x)\xi_k(x)\, dx\right\}_{i,j,k=1,\ldots,N}\right)\end{aligned} \tag{41}$$

where the multiple integral is over the domain

$$\mathcal{D} = \{a \le x_1 \le b,\, a \le x_2 \le b, \ldots, a \le x_N \le b\}$$

and $d\boldsymbol{x} = dx_1\, dx_2 \cdots dx_N$.

Proof: By rewriting the determinant using (37) we have

$$\int \cdots \int_{\mathcal{D}} |\boldsymbol{\Phi}(\boldsymbol{x})| \cdot |\boldsymbol{\Psi}(\boldsymbol{x})| \prod_{k} \xi_k(x_k)\, d\boldsymbol{x} \tag{42}$$

$$\begin{aligned}&= \int \cdots \int_{\mathcal{D}} \left[\sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{l} \Phi_{\sigma_l}(x_l)\right] \\ &\qquad \cdot \left[\sum_{\mu} \mathrm{sgn}(\mu) \prod_{m} \Psi_{\mu_m}(x_m)\right] \prod_{k} \xi_k(x_k)\, d\boldsymbol{x} \\ &= \sum_{\mu} \mathrm{sgn}(\mu) \sum_{\sigma} \mathrm{sgn}(\sigma) \\ &\qquad \int \cdots \int_{\mathcal{D}} \prod_{k} \Phi_{\sigma_k}(x_k)\Psi_{\mu_k}(x_k)\xi_k(x_k)\, d\boldsymbol{x} \\ &= \sum_{\mu} \mathrm{sgn}(\mu) \sum_{\sigma} \mathrm{sgn}(\sigma) \\ &\qquad \cdot \prod_{k} \int_a^b \Phi_{\sigma_k}(x_k)\Psi_{\mu_k}(x_k)\xi_k(x_k)\, dx_k\end{aligned} \tag{43}$$

and substituting the definition (39) into (43) gives (41). □

Corollary 1: Given two arbitrary $N \times N$ matrices $\boldsymbol{\Phi}(\boldsymbol{x})$ and $\boldsymbol{\Psi}(\boldsymbol{x})$ with ijth elements $\Phi_i(x_j)$ and $\Psi_i(x_j)$, and an arbitrary function $\xi(\cdot)$, the following identity holds:

$$\begin{aligned}&\int \cdots \int_{\mathcal{D}} |\boldsymbol{\Phi}(\boldsymbol{x})| \cdot |\boldsymbol{\Psi}(\boldsymbol{x})| \prod_{k=1}^{N} \xi(x_k)\, d\boldsymbol{x} \\ &\quad = N! \det\left(\left\{\int_a^b \Phi_i(x)\Psi_j(x)\xi(x)\, dx\right\}_{i,j=1,\ldots,N}\right)\end{aligned} \tag{44}$$

where the multiple integral is over the domain

$$\mathcal{D} = \{a \le x_1 \le b,\, a \le x_2 \le b,\, \ldots,\, a \le x_N \le b\}$$

and $d\boldsymbol{x} = dx_1\, dx_2 \cdots dx_N$.

Proof: This can be seen as a special case of Theorem 2 for degenerate tensors, where (40) applies.

Alternatively, (44) can be proven directly without Theorem 2 by writing the determinants explicitly using (37) as

$$\int \cdots \int_{\mathcal{D}} |\boldsymbol{\Phi}(\boldsymbol{x})| \cdot |\boldsymbol{\Psi}(\boldsymbol{x})| \prod_k \xi(x_k)\, d\boldsymbol{x} \tag{45}$$

$$= \int \cdots \int_{\mathcal{D}} \left[\sum_\sigma \operatorname{sgn}(\sigma) \prod_l \Phi_{\sigma_l}(x_l)\right] \cdot \left[\sum_\mu \operatorname{sgn}(\mu) \prod_m \Psi_{\mu_m}(x_m)\right] \prod_k \xi(x_k)\, d\boldsymbol{x} \tag{46}$$

$$= \sum_\mu \operatorname{sgn}(\mu) \sum_\sigma \operatorname{sgn}(\sigma) \int \cdots \int_{\mathcal{D}} \prod_k \Phi_{\sigma_k}(x_k)\Psi_{\mu_k}(x_k)\xi(x_k)\, d\boldsymbol{x} \tag{47}$$

$$= \sum_\mu \operatorname{sgn}(\mu) \sum_\sigma \operatorname{sgn}(\sigma) \cdot \prod_k \int_a^b \Phi_{\sigma_k}(x_k)\Psi_{\mu_k}(x_k)\xi(x_k)\, dx_k \tag{48}$$

$$= \sum_\mu \left| \left\{ \int_a^b \Phi_i(x)\Psi_j(x)\xi(x)\, dx \right\}_{i,j=1,\ldots,N} \right| \tag{49}$$

$$= N! \left| \left\{ \int_a^b \Phi_i(x)\Psi_j(x)\xi(x)\, dx \right\}_{i,j=1,\ldots,N} \right| \tag{50}$$

where we used (38) for passing from (48) to (49). □

To the best of the authors' knowledge, the preceding proof is original. However, Corollary 1 can be thought of as a continuous analog of the Cauchy-Binet formula and has been known in multivariate analysis as early as 1883 [31].

Corollary 2: Given two arbitrary $N \times N$ matrices $\boldsymbol{\Phi}(\boldsymbol{x})$ and $\boldsymbol{\Psi}(\boldsymbol{x})$ with ijth elements $\Phi_i(x_j)$ and $\Psi_i(x_j)$, and an arbitrary function $\xi(\cdot)$, the following identity holds:

$$\int \cdots \int_{\mathcal{D}_{\text{ord}}} |\boldsymbol{\Phi}(\boldsymbol{x})| \cdot |\boldsymbol{\Psi}(\boldsymbol{x})| \prod_{k=1}^N \xi(x_k)\, d\boldsymbol{x} = \det\left(\left\{ \int_a^b \Phi_i(x)\Psi_j(x)\xi(x)\, dx \right\}_{i,j=1,\ldots,N} \right) \tag{51}$$

where the multiple integral is over the domain $\mathcal{D}_{\text{ord}} = \{b \ge x_1 \ge x_2 \ge \ldots \ge x_N \ge a\}$.

Proof: Although $|\boldsymbol{\Phi}(\boldsymbol{x})|$ and $|\boldsymbol{\Psi}(\boldsymbol{x})|$ are not individually symmetric functions of $x_1, \ldots, x_N$ (a permutation of $x_1, \ldots, x_N$ corresponds to a permutation of the column of the matrices, and the determinant changes with the sign of the permutation), their product is clearly symmetric. Therefore, in Corollary 1, the integrand on the left side of (44) is a symmetric function of $x_1, \ldots, x_N$, which can be attributed for the scaling of a factor $N!$ from (44) to (51). □

Theorem 3: Given two arbitrary $N \times N$ matrices $\boldsymbol{\Phi}(\boldsymbol{x})$ and $\boldsymbol{\Psi}(\boldsymbol{x})$ with ijth elements $\Phi_i(x_j)$ and $\Psi_i(x_j)$, and two arbitrary functions $\xi(\cdot)$ and $\tilde{\xi}(\cdot)$, the following identity holds:

$$\int \cdots \int_{\mathcal{D}_{\text{ord}}} |\boldsymbol{\Phi}(\boldsymbol{x})| \cdot |\boldsymbol{\Psi}(\boldsymbol{x})| \prod_{m=1}^N \xi(x_m) \sum_{k=1}^N \tilde{\xi}(x_k)\, d\boldsymbol{x} \tag{52}$$

$$= \sum_{k=1}^N \det\left(\left\{ \int_a^b \Phi_i(x)\Psi_j(x)\xi(x)U_{k,j}\left(\tilde{\xi}(x)\right) dx \right\}_{i,j=1,\ldots,N} \right) \tag{53}$$

where the multiple integral is over the domain

$$\mathcal{D}_{\text{ord}} = \{b \ge x_1 \ge x_2 \ge \cdots \ge x_N \ge a\}$$

and the function $U_{k,j}(x)$ is defined by

$$U_{k,j}(x) \triangleq \begin{cases} x, & \text{if } k = j \\ 1, & \text{if } k \ne j. \end{cases} \tag{54}$$

Proof: We first consider the domain

$$\mathcal{D} = \{a \le x_1 \le b,\, a \le x_2 \le b,\, \ldots,\, a \le x_N \le b\}.$$

We can write

$$\int \cdots \int_{\mathcal{D}} |\boldsymbol{\Phi}(\boldsymbol{x})| \cdot |\boldsymbol{\Psi}(\boldsymbol{x})| \prod_m \xi(x_m) \sum_k \tilde{\xi}(x_k)\, d\boldsymbol{x} \tag{55}$$

$$= \int \cdots \int_{\mathcal{D}} \left[\sum_\sigma \operatorname{sgn}(\sigma) \prod_l \Phi_{\sigma_l}(x_l)\right] \cdot \left[\sum_\mu \operatorname{sgn}(\mu) \prod_i \Psi_{\mu_i}(x_i)\right] \prod_m \xi(x_m) \sum_k \tilde{\xi}(x_k)\, d\boldsymbol{x} \tag{56}$$

$$= \sum_\mu \operatorname{sgn}(\mu) \sum_\sigma \operatorname{sgn}(\sigma) \cdot \int \cdots \int_{\mathcal{D}} \prod_m \Phi_{\sigma_m}(x_m)\Psi_{\mu_m}(x_m)\xi(x_m) \sum_k \tilde{\xi}(x_k)\, d\boldsymbol{x} \tag{57}$$

$$= \sum_\mu \operatorname{sgn}(\mu) \sum_\sigma \operatorname{sgn}(\sigma) \cdot \sum_k \prod_m \int_a^b \Phi_{\sigma_m}(x_m)\Psi_{\mu_m}(x_m)\xi(x_m)U_{k,\mu_m}(\tilde{\xi}(x_m))\, dx_m \tag{58}$$

$$= \sum_k \sum_\mu \left| \left\{ \int_a^b \Phi_i(x)\Psi_j(x)\xi(x)U_{k,j}(\tilde{\xi}(x))\, dx \right\}_{i,j=1,\ldots,N} \right| \tag{59}$$

$$= N! \sum_k \left| \left\{ \int_a^b \Phi_i(x)\Psi_j(x)\xi(x)U_{k,j}(\tilde{\xi}(x))\, dx \right\}_{i,j=1,\ldots,N} \right| \tag{60}$$

where we used (38) for passing from (58) to (59). Then, we observe that even in this case, the integrand function in (52) is symmetric in the variables $\boldsymbol{x}$, which justifies the scaling of a factor $N!$ when integrating over $\mathcal{D}_{\text{ord}}$. □

Acknowledgment

The authors wish to thank A. Giorgetti for discussions about the capacity of correlated channels.

REFERENCES

[1] J. H. Winters, J. Salz, and R. D. Gitlin, "The impact of antenna diversity on the capacity of wireless communication system," *IEEE Trans. Commun.*, vol. 42, pp. 1740–1751, Feb./Mar./Apr. 1994.

[2] J. H. Winters, "On the capacity of radio communication systems with diversity in Rayleigh fading environment," *IEEE J. Select. Areas Commun.*, vol. JSAC-5, pp. 871–878, June 1987.

[3] G. J. Foschini, "Layered space-time architecture for wireless communication a fading environment when using multiple antennas," *Bell Labs Tech. J.*, vol. 1, no. 2, pp. 41–59, Autumn 1996.

[4] G. J. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Commun.*, vol. 6, no. 3, pp. 311–335, Mar. 1998.

[5] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, pp. 744–765, Mar. 1998.

[6] B. Hassibi and B. M. Hochwald, "High-rate codes that are linear in space and time," *IEEE Trans. Inform. Theory*, vol. 48, pp. 1804–1824, July 2002.

[7] N. Boubaker, K. B. Letaief, and R. D. Murch, "Performance of BLAST over frequency-selective wireless communication channels," *IEEE Trans. Commun.*, vol. 50, pp. 196–199, Feb. 2002.

[8] A. Lozano and C. Papadias, "Layered space–time receivers for frequency-selective wireless channels," *IEEE Trans. Commun.*, vol. 50, pp. 65–73, Jan. 2002.

[9] A. Zanella, M. Chiani, M. Z. Win, and J. H. Winters, "Symbol error probability of high spectral efficiency MIMO systems," in *Proc. Conf. Information Scences and Systems*, Princeton, NJ, Mar. 2002.

[10] X. Zhu and R. D. Murch, "Performance analysis of maximum likelihood detection in a MIMO antenna system," *IEEE Trans. Commun.*, vol. 50, pp. 187–191, Feb. 2002.

[11] E. Telatar, "Capacity of multi-antenna Gaussian channels," *Europ. Trans. Telecomm.*, vol. 10, no. 6, pp. 585–595, Nov.–Dec. 1999.

[12] T. L. Marzetta and B. M. Hochwald, "Capacity of a mobile multiple-antenna communication link in Rayleigh flat fading," *IEEE Trans. Inform. Theory*, vol. 45, pp. 139–157, Jan. 1999.

[13] P. J. Smith and M. Shafi, "On a Gaussian approximation to the capacity of wireless MIMO systems," in *Proc. IEEE Int. Conf. Communications*, vol. 1, New York, NY, May 2002, pp. 406–410.

[14] D.-S. Shiu, G. J. Foschini, M. J. Gans, and J. M. Kahn, "Fading correlation and its effect on the capacity of multielement antenna systems," *IEEE Trans. Commun.*, vol. 48, pp. 502–513, Mar. 2000.

[15] A. Abdi and M. Kaveh, "A space-time correlation model for multielement antenna systems in mobile fading channels," *IEEE J. Select. Areas Commun.*, vol. 20, pp. 550–560, Apr. 2002.

[16] S. Loyka and G. Tsoulos, "Estimating MIMO system performance using the correlation matrix approach," *IEEE Commun. Lett.*, vol. 6, pp. 19–21, Jan. 2002.

[17] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 1st ed. Cambridge, U.K.: Cambridge Univ. Press, 1990.

[18] R. A. Fisher, "Frequency distribution of the values of the correlation coefficient in samples from an indefinitely large population," *Biometrika*, vol. 10, pp. 507–521, 1915.

[19] J. Wishart, "The generalized product moment distribution in samples from a normal multivariate population," *Biometrika*, vol. 20A, pp. 32–52, 1928.

[20] ——, "Proofs of the distribution law of the second order moment statistics," *Biometrika*, vol. 35, pp. 55–57, 1948.

[21] A. T. James, "Distributions of matrix variates and latent roots derived from normal samples," *Ann. Math. Statist.*, vol. 35, pp. 475–501, 1964.

[22] M. Chiani, M. Z. Win, A. Zanella, R. K. Mallik, and J. H. Winters, "Bounds and approximations for optimum combining of signals in the presence of multiple co-channel interferers and thermal noise," *IEEE Trans. Commun.*, vol. 51, pp. 296–307, Feb. 2003.

[23] C. G. Khatri, "On the moments of traces of two matrices in three situations for complex multivariate normal populations," *Sankhya, The Indian J. Statist., Ser. A*, vol. 32, pp. 65–80, 1970.

[24] H. Gao, P. J. Smith, and M. V. Clark, "Theoretical reliability of MMSE linear diversity combining in Rayleigh-fading additive interference channels," *IEEE Trans. Commun.*, vol. 46, pp. 666–672, May 1998.

[25] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions wih Formulas, Graphs, and Mathematical Tables*. Washington, D.C.: U. S. Dept. Commerce, 1970.

[26] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 5th ed. San Diego, CA: Academic, 1994.

[27] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, 1st ed. Cambridge, U.K.: Cambridge Univ. Press, 1994.

[28] V. A. Aalo, "Performance of maximal-ratio diversity systems in a correlated Nakagami-fading environment," *IEEE Trans. Commun.*, vol. 43, pp. 2360–2369, Aug. 1995.

[29] J. Salz and J. H. Winters, "Effect of fading correlation on adaptive arrays in digital mobile radio," *IEEE Trans. Veh. Technol.*, vol. 43, pp. 1049–1057, Nov. 1994.

[30] P. D. Lax, *Linear Algebra*, 1st ed. New York: Wiley, 1996.

[31] M. C. Andréief, "Note sur une relation entre les intégrales définies des produits des fonctions," *Mém. de la Soc. Sci. de Bordeaux*, vol. 2, pp. 1–14, 1883.