On Optimum Combining of M-PSK Signals With Unequal-Power Interferers and Noise

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***Abstract*—In this letter, we derive a closed-form symbol-error probability expression for adaptive antenna array with optimum (or, equivalently, linear minimum mean-square error) combining. We consider coherent detection of M-ary phase-shift keying signals in the presence of unequal-power interferers and thermal noise. The analysis is based on our new results on the eigenvalues distribution of central Wishart matrices with correlation.**

***Index Terms*—Adaptive arrays, antenna diversity, cochannel interference, eigenvalues distribution, minimum mean-square error (MMSE) receivers, optimum combining (OC), Wishart matrices.**

I. Introduction

ADAPTIVE arrays using optimum combining (OC) can significantly improve the performance of wireless communication systems by weighting and combining the received signals to reduce fading effects and suppress interference [1]. The performance evaluation of OC is known to be difficult, especially if fading is taken into account for all the interfering signals, in addition to the desired signal [2]–[8].

Bit-error probability (BEP) expressions for OC in the presence of the single interferer were derived [1], [2], [9]. In [1], fading of the interferer was not taken into account explicitly, whereas [2] and [9] consider the case in which both the desired signal and a single interferer are subject to Rayleigh fading.

For the case of multiple equal-power interferers (EPI) in Rayleigh fading channels, upper bounds on the symbol-error probability (SEP) have been derived in [3]–[5], and closed-form BEP and SEP expressions are obtained more recently in [10] and [11] for binary phase-shift keying (BPSK) and [6]–[8] for M-ary phase-shift keying (M-PSK). With multiple interferers of arbitrary power, Monte Carlo simulation has been used to determine the BEP [1], and upper bounds on the BEP were derived [12], [13]. However, analytical results are not known for the OC of signals in the presence of multiple, uncorrelated, unequal-power interferers (UPI) as well as thermal noise in a Rayleigh fading environment.

In this letter, we derive a concise SEP expression for adaptive antenna array with optimum [or, equivalently, linear minimum mean-square error (MMSE)] combining. We consider coherent detection of M-PSK signals in the presence of multiple, uncorrelated UPI, as well as thermal noise in a flat Rayleigh fading environment. The analysis is based on our recent results on the eigenvalues distribution of complex Wishart matrices with correlation [7], [14].

II. System Description

The received signal at the $N_{\rm A}$-element array output consists of the desired signal, $N_{\rm I}$ interfering signals, and thermal noise. After matched filtering and sampling at the symbol rate, the array output vector at time k can be written as [4]

$$\mathbf{z}(k)=\sqrt{E_{\rm D}}\boldsymbol{c}_{\rm D}b_0(k)+\mathbf{z}_{\rm IN}(k) \tag{1}$$

with the interference-plus-noise term

$$\mathbf{z}_{\rm IN}=\sum_{j=1}^{N_I}\sqrt{E_j}\boldsymbol{c}_{{\rm I},j}b_j(k)+\mathbf{n}(k) \tag{2}$$

where $E_{\rm D}$ and E_j are the mean (over fading) energies of the desired and jth interfering signal, respectively, $\boldsymbol{c}_{\rm D}=[\boldsymbol{c}_{{\rm D},1},\ldots,\boldsymbol{c}_{{\rm D},N_{\rm A}}]^T$ and $\boldsymbol{c}_{{\rm I},j}=[\boldsymbol{c}_{{\rm I},j,1},\ldots,\boldsymbol{c}_{{\rm I},j,N_{\rm A}}]^T$ are the desired and jth interference-normalized propagation vectors, respectively, $b_0(k)$ and $b_j(k)$ are the desired and interfering data samples, respectively, and $\mathbf{n}(k)$ represents the additive noise.[1] Without loss of generality, we index the interferers such that $E_j\geq E_{j+1}$, for all $j=1,\ldots N_{\rm I}-1$.

We model $\boldsymbol{c}_{\rm D}$ and $\boldsymbol{c}_{{\rm I},j}$ as multivariate complex-valued Gaussian vectors having $\mathbb{E}\{\boldsymbol{c}_{\rm D}\}=\mathbb{E}\{\boldsymbol{c}_{{\rm I},j}\}=\mathbf{0}$ and $\mathbb{E}\{\boldsymbol{c}_{\rm D}\boldsymbol{c}_{\rm D}^{\dagger}\}=\mathbb{E}\{\boldsymbol{c}_{{\rm I},j}\boldsymbol{c}_{{\rm I},j}^{\dagger}\}=\mathbf{I}$, where $\mathbf{I}$ is the identity matrix. The interfering data samples, $b_j(k)$ for $j=1,\ldots,N_{\rm I}$, can be modeled as uncorrelated zero-mean random variables, and without loss of generality, $b_0(k)$ and $b_j(k)$ are assumed to have unit variance. The additive noise is modeled as a white Gaussian random vector with independent and identically distributed (i.i.d.) elements with $\mathbb{E}\{\mathbf{n}(k)\}=\mathbf{0}$ and $\mathbb{E}\{\mathbf{n}(k)\mathbf{n}^{\dagger}(k)\}=N_0\mathbf{I}$, where $N_0/2$ is the two-sided thermal noise power spectral density per antenna element.

III. Derivation of the SEP

The signal-to-interference-plus-noise ratio (SINR) at the output of the $N_{\rm A}$-element array with OC is given by [1]

$$\gamma=E_{\rm D}\boldsymbol{c}_{\rm D}^{\dagger}\mathbf{R}^{-1}\boldsymbol{c}_{\rm D} \tag{3}$$

[1]Throughout the letter, $(\cdot)^T$ is the transposition operator, and $(\cdot)^{\dagger}$ denotes conjugation and transposition.

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where $\mathbf{R}$ denotes the short-term covariance matrix of the disturbance $\mathbf{z}_{\rm IN}(k)$, conditioned on all interference propagation vectors. It is important to remark that $\mathbf{R}$ and, consequently, γ vary at the fading rate, which is assumed to be much slower than the symbol rate.

Using the approach similar to that of [3] and [8], it can be shown that

$$\gamma = E_{\rm D} \sum_{i=1}^{N_{\rm A}} \frac{|u_i|^2}{\tilde{\lambda}_i + N_0} \tag{4}$$

where $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{N_{\rm A}})$ are the eigenvalues of the $(N_{\rm A} \times N_{\rm A})$ random matrix $\tilde{\mathbf{R}} = \tilde{\mathbf{C}}_{\rm I}\mathbf{\Sigma}\tilde{\mathbf{C}}_{\rm I}^{\dagger}$. The $(N_{\rm A} \times N_{\rm I})$ matrix $\tilde{\mathbf{C}}_{\rm I}$ is composed of $N_{\rm I}$ normalized interference propagation vectors as columns

$$\tilde{\mathbf{C}}_{\rm I} \triangleq \begin{bmatrix} | & | & & | \\ \boldsymbol{c}_{{\rm I},1} & \boldsymbol{c}_{{\rm I},2} & \cdots & \boldsymbol{c}_{{\rm I},N_{\rm I}} \\ | & | & & | \end{bmatrix} \tag{5}$$

and the matrix $\mathbf{\Sigma} = \operatorname{diag}\{E_1, E_2, \ldots, E_{N_{\rm I}}\}$ takes into account the interferers' power levels. The vector $\mathbf{u} = \mathbf{U}^{\dagger}\boldsymbol{c}_{\rm D} = [u_1, \ldots, u_{N_{\rm A}}]^T$ has the same distribution as $\boldsymbol{c}_{\rm D}$, since $\mathbf{U}$ represents a unitary transformation. Note that even the eigenvalues vary at the fading rate.

Assuming $N_{\rm I} \le N_{\rm A}$, it is simple to show that the SEP can be written as [8]

$$\begin{aligned} P_e &= \mathbb{E}_{\tilde{\lambda}}\{P_{e|\tilde{\lambda}=\mathbf{x}}\} \\ &= \int_0^{\infty} \cdots \int_{x_3}^{\infty}\int_{x_2}^{\infty} P_{e|\tilde{\lambda}=\mathbf{x}} f_{\tilde{\lambda}}(\mathbf{x}) dx_1 dx_2 \ldots dx_{N_{\rm I}} \end{aligned} \tag{6}$$

where $P_{e|\tilde{\boldsymbol{\lambda}}=\mathbf{x}}$ is the conditional SEP, conditioned on a given realization of $\tilde{\boldsymbol{\lambda}} = \mathbf{x}$, and $f_{\tilde{\boldsymbol{\lambda}}}(\mathbf{x})$ is the joint probability density function (pdf) of the $N_{\rm I}$ nonzero ordered eigenvalues of $\tilde{\mathbf{R}}$. The expression (6) can be cumbersome to evaluate, as it requires the evaluation of nested $N_{\rm I}$-fold integrals. However, we show how this expression can be simplified to a concise expression, given in the following theorem.

Theorem 1: The exact SEP expression for coherent detection of M-PSK with OC is

$$P_e = \frac{K_{\rm corr}}{\pi} \int_0^{\Theta} A(\theta) \det \mathcal{G}\, d\theta \tag{7}$$

where $\Theta \triangleq \pi(M-1)/M$, $c_{\rm MPSK} \triangleq \sin^2(\pi/M)$

$$A(\theta) = \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\rm MPSK}\frac{E_{\rm D}}{N_0}} \right]^{N_{\rm A}-N_{\rm I}} \tag{8}$$

and

$$K_{\rm corr} = K \cdot \zeta_{N_{\rm I}} \cdot \frac{|\mathbf{\Sigma}|^{-N_{\rm A}}}{\left|\mathbf{V}\left(-E_i^{-1}\right)\right|} \tag{9}$$

where $|\mathbf{A}| = \det \mathbf{A}$ denotes the determinant of $\mathbf{A}$.

The constant K is given by

$$K = \frac{\pi^{N_{\rm I}(N_{\rm I}-1)}}{\tilde{\Gamma}_{N_{\rm I}}(N_{\rm A})\tilde{\Gamma}_{N_{\rm I}}(N_{\rm I})} \tag{10}$$

with

$$\tilde{\Gamma}_{N_{\rm I}}(n) = \pi^{\frac{N_{\rm I}(N_{\rm I}-1)}{2}} \prod_{i=1}^{N_{\rm I}} (n-i)! \tag{11}$$

$$\zeta_{N_{\rm I}} \triangleq \prod_{j=1}^{N_{\rm I}} (j-1)! \tag{12}$$

and $\mathbf{V}(x_j) \triangleq \{x_j^{i-1}\}_{i,j=1,\ldots,N_{\rm I}}$ is a Vandermonde matrix. The elements of the $(N_{\rm I} \times N_{\rm I})$ matrix $\mathcal{G} = \{g_{i,j}\}_{i,j=1,\ldots,N_{\rm I}}$ in (7) are given by

$$\begin{aligned} g_{i,j} = &\left(N_0 + \frac{c_{\rm MPSK}E_{\rm D}}{\sin^2\theta}\right)^{N_{\rm A}-N_{\rm I}+j-1} e^{\frac{N_0\sin^2\theta + c_{\rm MPSK}E_{\rm D}}{E_i\sin^2\theta}} \\ &\times (N_{\rm A}-N_{\rm I}+j-1)! \\ &\times \Bigg[(N_{\rm A}-N_{\rm I}+j)\left(N_0 + \frac{c_{\rm MPSK}E_{\rm D}}{\sin^2\theta}\right) \\ &\quad \times \Gamma\left(N_{\rm I}-N_{\rm A}-j, \frac{N_0\sin^2\theta + c_{\rm MPSK}E_{\rm D}}{E_i\sin^2\theta}\right) \\ &\quad + N_0\Gamma\left(N_{\rm I}-N_{\rm A}-j+1, \vphantom{\frac{a}{b}}\right. \\ &\qquad \left.\frac{N_0\sin^2\theta + c_{\rm MPSK}E_{\rm D}}{E_i\sin^2\theta}\right)\Bigg] \end{aligned} \tag{13}$$

where $\Gamma(a,x)$ is the incomplete Gamma function [15, p. 949, eq. (8.350)].

Proof: For coherent detection of M-PSK signals, the conditional SEP is given by [3], [5], [8]

$$P_{e|\tilde{\lambda}=\mathbf{x}} = \frac{1}{\pi}\int_0^{\Theta} A(\theta) \prod_{i=1}^{N_{\rm I}} \left[\frac{\sin^2\theta}{\sin^2\theta + c_{\rm MPSK}\frac{E_{\rm D}}{x_i+N_0}}\right] d\theta \tag{14}$$

where $A(\theta)$ is defined in (8). The expression for the joint pdf of the eigenvalues in [8] is for the case of EPI, and cannot be used here to analyze the case of interest, namely UPI.

Since $N_{\rm I} \le N_{\rm A}$, it is possible to show that the $(N_{\rm I} \times N_{\rm I})$ matrix $\mathbf{W} = \mathbf{\Sigma}^{1/2}\tilde{\mathbf{C}}_{\rm I}^{\dagger}\tilde{\mathbf{C}}_{\rm I}\mathbf{\Sigma}^{1/2}$ is a full rank (with probability 1) central Wishart matrix, whose eigenvalues are the same as the nonzero eigenvalues of $\tilde{\mathbf{R}}$. The joint pdf of the (real) ordered eigenvalues $\tilde{\lambda}_1 \ge \tilde{\lambda}_2 \ge \ldots \ge \tilde{\lambda}_{N_{\rm I}}$ of $\mathbf{W}$ has been derived in [16] as

$$\begin{aligned} f_{\tilde{\lambda}}(x_1, \ldots, x_{N_{\rm I}}) = &K|\mathbf{\Sigma}|^{-N_{\rm A}} {}_0\tilde{F}_0(-\mathbf{\Sigma}^{-1}, \mathbf{W}) \\ &\times |\mathbf{W}|^{N_{\rm A}-N_{\rm I}} \prod_{i<j}^{N_{\rm I}} (x_i - x_j)^2. \end{aligned} \tag{15}$$

In (15), ${}_0\tilde{F}_0(\mathbf{A}, \mathbf{B})$ is a hypergeometric function of Hermitian matrix arguments, whose definition is given in [16, eq. (88)] in terms of *zonal polynomials*. Since these are quite difficult to manage, (15) is not of practical interest.

Fortunately, a more tractable expression for the joint pdf of the eigenvalues of the central Wishart matrix $\mathbf{W}$, with an *arbitrary* correlation matrix $\mathbf{\Sigma}$, was recently obtained in [14] as

$$f_{\tilde{\lambda}}(x_1,\ldots,x_{N_\mathrm{I}}) = K_\mathrm{corr}\left|\left\{x_j^{i-1}\right\}_{i,j}\right|\cdot\left|\left\{e^{-\frac{x_j}{\sigma_i}}\right\}_{i,j}\right| \times\prod_{j=1}^{N_\mathrm{I}} x_j^{N_\mathrm{A}-N_\mathrm{I}} \quad (16)$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{N_\mathrm{I}} \geq 0$ are the ordered eigenvalues of $\mathbf{\Sigma}$, and K_corr was defined previously in (9). Note that for the special case of our interest, $\sigma_i = E_i$.

Now, substituting (14) and (16) into (6), the expression for the SEP becomes

$$P_e = \frac{K_\mathrm{corr}}{\pi}\int_0^{\Theta} A(\theta)\int_0^{\infty}\cdots\int_{x_3}^{\infty}\int_{x_2}^{\infty}\left|\left\{x_j^{i-1}\right\}\right|\cdot\left|\left\{e^{-\frac{x_j}{\sigma_i}}\right\}\right| \times\prod_{j=1}^{N_\mathrm{I}}\left(\frac{x_j^{N_\mathrm{A}-N_\mathrm{I}}}{1+\frac{c_\mathrm{MPSK}}{\sin^2\theta}\frac{E_\mathrm{D}}{x_j+N_0}}\right)dx_1dx_2\ldots dx_{N_\mathrm{I}}d\theta. \quad (17)$$

This expression can be further simplified using the following identity proved in [14]. Given two arbitrary $N\times N$ matrices $\mathbf{\Phi}(\mathbf{x})$ and $\mathbf{\Psi}(\mathbf{x})$ with ijth elements $\Phi_i(x_j)$ and $\Psi_i(x_j)$, and an arbitrary function $\xi(\cdot)$, the following identity holds:

$$\int\cdots\int_{\mathcal{D}_{ord}}|\mathbf{\Phi}(\mathbf{x})|\cdot|\mathbf{\Psi}(\mathbf{x})|\prod_{i=1}^{N}\xi(x_i)d\mathbf{x} = \det\left(\left\{\int_a^b \Phi_i(x)\Psi_j(x)\xi(x)dx\right\}_{i,j=1\ldots,N}\right) \quad (18)$$

where the multiple integral is over the domain $\mathcal{D}_{ord} = \{b \geq x_1 \geq x_2 \ldots \geq x_N \geq a\}$ and $d\mathbf{x} = dx_1dx_2\cdots dx_N$.

Using (18), we easily obtain (7), with the ijth elements of the matrix $\mathcal{G}$ given by

$$g_{i,j} = \int_0^{\infty} x^{N_\mathrm{A}-N_\mathrm{I}+j-1}e^{-\frac{x}{\sigma_i}}\psi(x,\theta)dx \quad (19)$$

with

$$\psi(x,\theta) \triangleq \frac{x+N_0}{x+N_0+\frac{c_\mathrm{MPSK}E_\mathrm{D}}{\sin^2\theta}}. \quad (20)$$

Finally, by solving (19), we obtain (13). This completes the proof of the theorem. □

Thus, *Theorem 1* provides a concise expression for the efficient performance evaluation for OC of signals in the presence of multiple, uncorrelated UPI and thermal noise in Rayleigh fading.

As an example, we provide in Fig. 1 the SEP for quaternary phase-shift keying (QPSK) with $N_\mathrm{A} = 3$ and 5, $N_\mathrm{I} = 3$. The signal-to-noise ratio (SNR) is defined as E_D/N_0, the signal-to-interference ratio (SIR) as $E_\mathrm{D}/\sum_{j=1}^{N_\mathrm{I}}E_j$, and $\Gamma_j = E_\mathrm{D}/E_j$. The figure shows a comparison between the EPI and UPI, both cases with the same value of SIR $= 5$ dB. It can be seen that the EPI result in a higher SEP, regardless of the number of antenna elements considered.

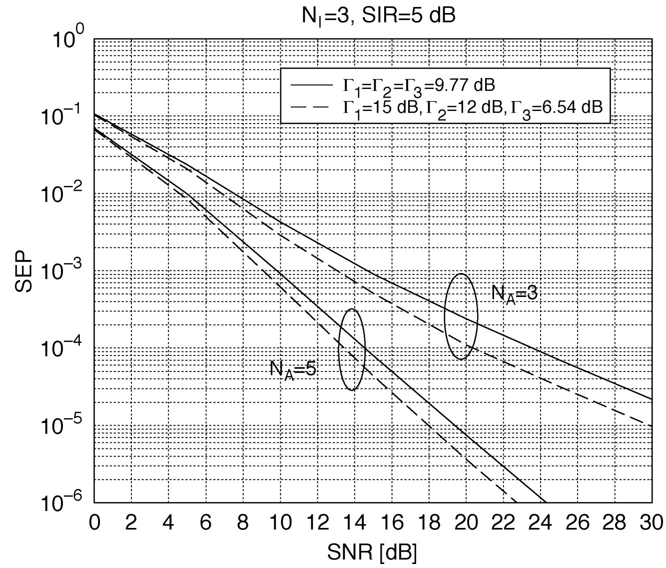

Fig. 1. SEP as a function of SNR for $N_\mathrm{A} = 3$ and 5, $N_\mathrm{I} = 3$, SIR $= 5$ dB, and QPSK; comparison between EPI and UPI.

IV. Conclusions

We derived a concise closed-form expression for efficient evaluation of the SEP for coherent detection of M-PSK using OC in the presence of multiple, uncorrelated, unequal-power cochannel interferers and thermal noise in a flat Rayleigh fading environment. The analysis is based on our new results on the eigenvalues distribution of central Wishart matrices with correlation [14].

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